These negative energy states lead to problems when one tries to use Dirac equation to describe highly relativistic single particle dynamics (see homework problem 2). They can only be given a proper treatment in a context of a many body theory as described by relativistic fields. So let's consider a classical field theory of \( \Psi_\alpha(x,t), \Psi^\dagger_\alpha(x,t) \) \( \alpha = 1, \ldots 4 \) as 4 complex classical fields with a classical Lagrangian:

\[
L(x,t) = \frac{i}{2} \left[ \overline{\Psi} \gamma^\mu (\partial_\mu \Psi) - (\partial_\mu \overline{\Psi}) \gamma^\mu \Psi \right] - m \overline{\Psi} \Psi
\]

where we have introduced the following notation:

\[
\overline{\Psi}_\alpha = \Psi^*_\beta \gamma_\beta \equiv \gamma^+ \gamma^0 = (\gamma^0 \Psi)^+ = [\gamma^0]_{\alpha\beta} \Psi_\beta
\]

Note \( \gamma_{\alpha\beta} = \delta_{\alpha\beta} \) is real. \( \overline{\Psi}_\alpha \Psi^\dagger_\alpha \)
We then have from the equations of motion, for instance

\[ \frac{\partial \mathcal{L}}{\partial \dot{\Psi}} - \frac{\partial}{\partial x} \frac{\partial \mathcal{L}}{\partial (\partial_x \Psi)} = 0 = \frac{i}{2} \bar{\Psi} \gamma^\mu \partial_\mu \Psi - m \Psi + \frac{i}{2} \partial_\mu \bar{\Psi} \gamma^\mu \Psi = 0 \]

and similarly when varying with respect to \( \bar{\Psi} \).

Thus the equations of motion are the Dirac equations:

\[ (i \gamma^\mu \partial_\mu - m) \Psi = 0 \]

[Dirac equation]

\[ \bar{\Psi} (i \gamma^\mu \partial_\mu + m) = 0 \]

We can now proceed as we did for the scalar field.

From the invariance of \( \mathcal{L} \) under space-time, symmetries derive the classical expressions for momentum, \( P_i \), energy \( H \), angular momentum \( \mathcal{J} \), in terms of the fields \( \Psi \) and \( \bar{\Psi} \).

These are given by:

\[ \Theta^{\mu \nu} = \frac{i}{2} \bar{\Psi} \gamma^\mu \gamma^\nu \Psi \]

\[ P_i = \int d^4x \Theta^{0i} = \int d^3x \frac{i}{2} \bar{\Psi} \gamma^0 \gamma^i \Psi \]

\[ H = \int d^4x \Theta^{00} = \int d^3x \frac{i}{2} \bar{\Psi} \gamma^0 \gamma^0 \Psi \]

\[ \mathcal{J} = \int d^4x \left[ x^j \Theta^{0k} - x^k \Theta^{0j} + \frac{i}{2} \bar{\Psi} \gamma^j \gamma^k \Psi \right]_{i,j,k = \text{cyclic permutations}} \]
The next step would be to calculate canonical momenta and impose commutation relations to quantize the system.

Note that the canonical momentum defines through \( \frac{\partial H}{\partial p_i} \) and \( \frac{\partial H}{\partial q_j} \) are proportional to the fields themselves and not their derivatives, thus canonical quantization will involve just the fields and their \( \partial_\mu \) and \( \partial^\mu \) parts to be obtained from equations of motion. As before we want to express \( H \) and \( \bar{\Psi} \) quantum fields in such a way that the Hamiltonian, \( \hat{H} \), and \( \hat{\bar{\Psi}} \) operators are diagonal:

\[
H = \int d^3x \bar{\Psi} \left[ -i \gamma_\mu \partial_\mu + m \right] \Psi \quad \text{from (*) and eq. to motion (4,*)}
\]

Let's try

\[
\Psi(x) = \int \frac{d^3p}{(2\pi)^3} \sum_{\alpha=1,2} \left[ b_\alpha(p) u(\bar{\Psi}) e^{\frac{iGp}{\hbar}} + b^*_\alpha(p) \bar{u}(\Psi) e^{-\frac{iGp}{\hbar}} \right]
\]

\[
\bar{\Psi}(x) = \int \frac{d^3p}{(2\pi)^3} \sum_{\alpha=1,2} \left[ b^*_\alpha(p) \bar{u}(\bar{\Psi}) e^{-\frac{iGp}{\hbar}} + b_\alpha(p) \bar{u}(\Psi) e^{\frac{iGp}{\hbar}} \right]
\]

\[
E_{\Psi} = \mathcal{E}(p) = \sqrt{m^2 + p^2}
\]
Where $b, b^\dagger, d, d^\dagger$ are some operators, $\lambda, \mu$ are the solutions of the Dirac equation. Note we have not yet imposed commutation relations between $\Psi_\alpha$ and $\bar{\Psi}_\alpha$ in the quantum theory, so we do not know what relations the operators $b, b^\dagger, d, d^\dagger$ should satisfy. The reason for that will become obvious soon.

Now calculate $H$:

$$H = \sum_{\alpha, \beta = 1}^3 \int d^3 \mathbf{x} \frac{\hbar^3}{(\hbar^3)^3} \left[ b^\dagger_{\alpha}(\mathbf{x}) b_{\beta}(\mathbf{x}) + d^\dagger_{\alpha}(\mathbf{x}) d_{\beta}(\mathbf{x}) \right] \hbar \mathbf{x} \cdot \mathbf{v} + \mu \left[ b_{\alpha}(\mathbf{x}) \Psi_{\alpha}(\mathbf{x}, p) + d^\dagger_{\alpha}(\mathbf{x}) \bar{\Psi}_{\alpha}(\mathbf{x}, p) \right] \hbar \mathbf{x} \cdot \mathbf{p}$$

$$= \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \mathbf{E}(\mathbf{p}) \left[ b^\dagger_{\alpha}(\mathbf{p}) b_{\alpha}(\mathbf{p}) - d^\dagger_{\alpha}(\mathbf{p}) d_{\alpha}(\mathbf{p}) \right]$$

where we have used:

$$\mathbf{v} + \mu \mathbf{p} = 2 \mathbf{E}(\mathbf{p})$$

$$\mathbf{v} - \mu \mathbf{p} = -2 \mathbf{E}(\mathbf{p})$$

$$\mathbf{v} + \mu \mathbf{p} = \mathbf{v} - \mu \mathbf{p}$$

$$\mathbf{v} = 0$$
Now suppose the commutational relations for the fields were such that for the operators $\alpha^\dagger$, $\alpha^\dagger$ would imply $[d\alpha(h), d\alpha^\dagger(h')] = (2\xi)^3 \delta^3(h-h') d\alpha$ i.e. just like before in the case of scalar theory.

Then

$$H = \sum_{\alpha = 1, 2} \int \frac{d^3 \mathbf{r}}{(2\pi)^3} \text{Tr} \left[ \langle \hat{h}(\mathbf{r}) \rangle \left[ b^\dagger_{\alpha}(h') b_{\alpha}(h) - d^\dagger_{\alpha}(h') d_{\alpha}(h) \right] \right] + \text{const}$$

and states $|\mathbf{h}, \alpha\rangle = d^\dagger_{\alpha}(h)|0\rangle$ where $|0\rangle$ is the vacuum state. $|0\rangle$: $b_{\alpha}(h)|0\rangle = d_{\alpha}(h)|0\rangle = 0$ would have negative energies and the Hamiltonian would be unbound from below $\Rightarrow$ total collapse!!

To cure this we have to have anticommutation relations between the operators in the fermion case rather than commutation relations:

$$\{d\alpha(h), d_{\alpha'}(h')\} = \delta_{\alpha\alpha'} \delta^3(h-h') + \delta_{\alpha\alpha'}(h) d_{\alpha}(h)$$

$$= (2\xi)^3 \delta^3(h-h') d\alpha$$
Which translates into:

\[ H = \sum_{\alpha = 1, 2} \frac{\hbar^2}{(2\pi)^3} E(\alpha) \left[ b_\alpha^+ (h) b_\alpha (h) \right] + \text{c.c.} + \text{drop.} \]

Similarly for momentum operators:

\[ p_i = \sum_{\alpha = 1, 2} \int \frac{d^3 \hbar}{(2\pi)^3} \ h^i \left[ b_\alpha^+ (h) b_\alpha (h) \right] \]

and total angular momentum operator:

\[ J_i = \int \frac{d^3 \hbar}{(2\pi)^3} \left[ b_\alpha^+ (h) \left( L_r^r \xi + S_{r\alpha}^r \right) b_\alpha (h) \right] + \text{c.c.} + \text{drop.} \]

\[ L^i = -i \hbar \frac{\partial}{\partial h^i} \]  

\[ = \left[ \vec{x} \times \vec{p} \right]^i \]

\[ \text{and} \quad S^i_{\alpha \beta} = \frac{1}{2} \sigma^i_{\alpha \beta} \text{ spin operators.} \]

Here the helicity indices defined in the following way.