Furthermore we see that from

\[ \vec{E} = 0 \] being a gauge independent constraint

we cannot separate uniquely \( \frac{\partial}{\partial t} A^0 \) and \( \partial A^0 \).

Instead in order to derive the complete set of independent dynamical degrees of freedom we should use the gauge symmetry to eliminate redundant variables.

This can be done in many ways.

1) Suppose we require \( A^0(x) = 0 \) we'll call the \( A^0 \) field in this case \( v^0 \).

If we start from \( A^0(x,t) \neq 0 \) then a gauge transformation

\[ A^0 \rightarrow V^0 = A^0 + e^0 \frac{\partial}{\partial t} \Theta(x,t) = 0 \]

\[ \Rightarrow \frac{\partial}{\partial t} \Theta(x,t) = -e A^0(x,t) \Rightarrow \Theta(x,t) = -e \int_0^t A^0(x,t') dt' + f(x) \]

which is a unique solution modulo a function, \( f(x) \), independent of \( t \). Thus we still have gauge freedom but with \( t \)-independent transformations.
Now we have:

\[ \Pi^i = -e^i = (\frac{2}{j} A^i + \frac{2}{j} x^i A^0) \Rightarrow (\frac{2}{j} V^i + \frac{2}{j} x^i V^0) = \frac{2}{j} V^i \]

The Hamiltonian looks like:

\[ H = \Pi^i \dot{V}^i - L = \Pi^2 - \frac{1}{2} \Pi^2 + \frac{i}{2} \dot{B}^2 = \frac{1}{2} (\Pi^2 + \dot{B}^2) \]

And we can quantize by requiring:

\[ [\Pi^i (\vec{x}, t), V^j (\vec{y}, t)] = -i \delta^i_j (\vec{x} - \vec{y}) \]

at all times. The condition \( \overline{\partial} \Pi = -\overline{\partial} E = 0 \) can only be imposed on states:

\[ \overline{\partial} \Pi (\vec{x}, t) \mid \text{(physical)} = 0 \]

rather than as an operator equation.

This condition manifests the residual (time independent) gauge symmetry i.e., it states that only gauge invariant physical states should be considered.

Indeed one can show that \( \overline{\partial} \Pi \) is the
generals of the residual transformations:

\[ V_j (x, t) \rightarrow V_j' (x, t) = V_j (x, t) - \frac{1}{\epsilon} \nabla_j \Theta (x) \]

function of \( \tilde{r} \) only

Now define:

\[ V_j (x, t) = e \left[ - \frac{1}{\epsilon} \int d^2 y \frac{\partial^2}{\partial^2 (y)} \Theta (x, t) + \frac{1}{\epsilon} \int d^2 y \frac{\partial}{\partial y} \frac{\partial}{\partial y} \Theta (x, t) \right] e^{-\epsilon (\cdot)} \]

\[ = e^{\frac{i}{\epsilon} \int d^2 y \frac{\partial}{\partial y} \Theta (x, t) \frac{\partial}{\partial y} \Theta (x, t)} e^{-\epsilon (\cdot)} \]

\[ = \left( e^{\frac{i}{\epsilon} \int d^2 y \frac{\partial}{\partial y} \Theta (x, t)} \right) e^{-\epsilon (\cdot)} \]

\[ = - \frac{1}{\epsilon} \frac{\partial}{\partial x} \nabla_j \Theta (x) \]

\[ \Rightarrow V_j (x, t) = V_j (x, t) - \frac{1}{\epsilon} \nabla_j \Theta (x), \quad V_j (x, t) = V_j (x, t) \]

so indeed \( \tilde{r} (x, t) \) is the generating of local...
The $A^0 = 0$ gauge fixing is known as the Ward gauge.

**Coulomb gauge**

Alternatively, we can set $\nabla \bar{A}(\vec{x}, t) = 0$

Again let's call the field after gauge fixing $V^\mu$, $A^\mu \rightarrow V^\mu$

$x(\vec{x}, t) = \nabla \bar{A}(\vec{x}, t)$

Then

$\nabla \bar{V}(\vec{x}, t) = \nabla \bar{A}(\vec{x}, t) - \frac{\lambda}{\epsilon} \nabla^2 \Theta(x, t) = x(\vec{x}, t) - \frac{\lambda}{\epsilon} \nabla^2 \Theta(x, t)$

means $\nabla^2 \Theta(x, t) = \epsilon x(\vec{x}, t)$

i.e. $\Theta$ is a solution of Poisson's eq. which has a unique solution provided we impose "physical" boundary conditions at $|\vec{x}| \rightarrow \infty$

Now

$\Pi^i = -E^i = \left( \frac{\partial}{\partial t} A^i + \frac{2}{\epsilon} j^i A^0 \right) \rightarrow \left( \frac{\partial}{\partial t} V^i + \frac{2}{\epsilon} j^i V^0 \right)$

and

$\nabla \Pi = -\nabla \bar{E} = \left( \frac{\partial}{\partial t} \nabla / V + \nabla^2 V^0 \right)$

The condition $\nabla \bar{E} = 0$ will now constrain $V^0$

$\nabla^2 V^0 = 0$
Note that now there is no residual freedom in gauge transformations, since $\Theta(x,t)$ has been uniquely determined by $\kappa(x,t)$ for all times.

What about Hamiltonian:

$H = \Pi^i \frac{dv^i}{dt} - L$

It is noted that any function of $x$ can be decomposed into transverse and longitudinal piece:

$\Pi^i(x,t) = \Pi^i_T(x,t) + \Pi^i_L(x,t) = \left( \Pi^i - \frac{\partial^2 \Phi}{\partial x^i} \right) + \frac{\partial^i}{\partial x^i} \left( \frac{\partial^2 \Phi}{\partial x^i} \right)$

such that $\overrightarrow{\nabla} \Pi^i_T(x,t) = 0$ and $\overrightarrow{\nabla} \times \Pi^i_L(x,t) = 0$

For the Hamiltonian we get:

$H = \int d^4x \phi \left( \Pi^i_T + \Pi^i_L \right) \frac{d}{dt} V^i_T - \phi \left( \Pi^i_T \frac{d}{dt} V^i_T \right) - L$

From $\Pi^i = \frac{d}{dt} V^i + \frac{1}{2} \overrightarrow{\nabla} \phi \left( \frac{1}{2} \overrightarrow{\nabla} \phi \right) V^i$,

$\Rightarrow \quad H = \Pi^i_T \Pi^i_T - \phi \left( \Pi^i_T \Pi^i_T - \frac{1}{2} \Pi^i_T \Pi^i_T + \frac{1}{2} \Pi^i_T \Pi^i_T \right) + \frac{1}{2} \phi \left[ \frac{1}{2} \Pi^i_T \Pi^i_T \right]$

$H = \frac{1}{2} \left( \Pi^2_T + B^2 \right)$

How det of $B$ only

$V_T$ contributes
Now we quantize Coulomb gauge by imposing

\[ [\Pi_T^i(x,t), \Pi_T^j(y,t)] = -i \left( \delta_{ij} - \frac{\partial_i \partial_j}{D^2} \right) \delta^3(x-y) \]

Summary:

Coulomb gauge $\Rightarrow$ from original $A^i$, $4$ degrees of freedom after fixing gauge freedom we have obtained a consistent theory of two degrees of freedom

$\Rightarrow$ transverse part of the $A$ field. No constraint (Coulomb law) is left to be imposed. The $A^0$ component is determined by $\nabla^2 A^0(x,t) = 0$. If matter fields (charges) were present this would change to

$\nabla^2 A^0(x,t) = e \rho(x,t)$

$\Rightarrow$ charge density.
In Weyl gauge the theory still involves a physical degree of freedom (3-component A potential). However, the residual gauge invariance imposed on physical states \( \vec{\Pi}(\vec{x}, t) |_{\text{phys}} \) eliminates dependence on longitudinal component (just like Coulomb gauge).

**Solution of Eq.** (Coulomb gauge).

\[
\mathcal{H} = \frac{1}{4} \int d^3 x \left[ \vec{\Pi}^2(\vec{x}) + \vec{B}^2(\vec{x}) \right] \quad \vec{B} = (\vec{\nabla} \times \vec{A})
\]

Express

\[
\vec{A}(\vec{x}) = \sum_{\lambda} \int \frac{d^3 p}{(2\pi)^3} \left[ \frac{\epsilon^2}{(2\mu)^3} \frac{\epsilon^{*(\lambda)}(\vec{p}, \vec{x})}{\alpha(\vec{p}, \vec{x})} \alpha^{*(\lambda)}(\vec{p}, \vec{x}) \sqrt{\frac{m^2}{2}} \right]
\]

\[
\vec{\Pi}(\vec{x}) = -i \sum_{\lambda} \int \frac{d^3 p}{(2\pi)^3} \left[ \frac{m^2}{2} \right]
\]

\[
\vec{\nabla} \vec{A}(\vec{x}) = 0 \Rightarrow \vec{\nabla} \vec{\epsilon}(\vec{u}, \vec{x}) = 0 \quad \text{thus there are only two linearly independent vectors} \quad \vec{\epsilon}(\vec{u}, \vec{v})
\]

\( \vec{e}(\vec{u}, \vec{v}) \)
Helicity basis:
\[\tilde{e}(\Omega, \pm 1) = \pm \frac{1}{\hbar} \left[\tilde{e}(\Omega, 1) \pm \epsilon \tilde{e}(\Omega, 2)\right]\]

\[\left[\Pi^j_{T}(\mathbf{x}), A^j_{+}(\mathbf{y})\right] = -i \left(\delta^j - \frac{\partial^j}{\partial x^j}\right) \delta^{3}(\mathbf{x} - \mathbf{y})\]

\[\Rightarrow \left[\alpha(x', \xi'), \alpha^{+}(x, \xi)\right] = i(2\hbar)^3 \delta(x - x') \delta(\xi - \xi')\]

\[H = \frac{1}{2} \sum_{\Omega} \int \frac{d^{3}h}{(2\pi)^3} \frac{\omega^{2}(\Omega) + k^{2}}{2 \omega(\Omega)} \left[\alpha^{+}(h, \xi) \alpha(h, \xi/)\right]\]

\[+ \sum_{x} \int \frac{d^{3}h}{(2\pi)^3} \frac{\omega^{2}(\Omega) - k^{2}}{2 \omega(\Omega)} \left[\alpha^{+}(h, \xi) \alpha^{+}(h, \xi/) + \alpha(h, \xi) \alpha(h, \xi/)\right]\]

\[+ \text{constant}\]

\[\Rightarrow \text{take } \omega(\Omega) = k \text{ and have a diagonal, harmonic oscillator Hamiltonian:}\]

\[H = \sum_{\Omega} \int \frac{d^{3}h}{(2\pi)^3} \frac{\omega(\Omega)}{2} \left[\alpha^{+}(h, \xi) \alpha(h, \xi/) + \alpha(h, \xi) \alpha^{+}(h, \xi/)\right]\]