The Schwinger model

Let's try to understand the content of TDA and RPA by solving a simple model. Take the nucleon here to be spin-isospin independent. Both $L, S$, and $T$ are therefore good quantum numbers. We therefore should use an occupied basis $d, g, h, s, -\frac{1}{2}, m, \uparrow, \downarrow$ rather than the coupled one $d, g, h, s, -\frac{1}{2}, m, \uparrow, \downarrow$.

Let's explicitly carry out reduction of both $L - S$ states of $J = 1/2, 1, 3/2$ in turn first.

1) The ground state of even-even nuclei corresponds to closed shells with $S = L = 0$.

2) $d, g, h, s, -\frac{1}{2}, m, \uparrow, \downarrow$ where $m = \pm \frac{1}{2}, m, \uparrow, \downarrow$.

Instead of tracing back from $j - j$ coupling to $L - S$ coupling, it will be simpler to work with TDA starting from a slightly different conventional basis.

$$C_{a} = \Theta(F - F) d_{a} + \Theta(F - d_{a}) d_{a} d_{\uparrow} B_{-a}$$

$$\gamma_{a} = (-)^{\frac{1}{2} - m_{a} \frac{1}{2} - m_{a}} \frac{1}{2} - m_{a}$$
Let's define the TDA wave functions states of good $L, S, T$:

\[
X_{LST}^{(m)}(ab) \equiv \frac{1}{\sqrt{|\Delta T_{ab}|}} \left\langle \lambda_l \mu_l \mu_p | \lambda_a \mu_b \right\rangle \left\langle \frac{1}{2} \omega_{1a} \frac{1}{2} \omega_{2b} \right| \frac{1}{2} \omega_{1p} \left| \frac{1}{2} \omega_{2m} \right\rangle \chi^{(m)} \chi^{(n)}
\]

axioms:

\[
\Delta T_{ab} - \Delta T_{ba} = 0, \omega_{1a} - \omega_{2b}, \omega_{1b} - \omega_{2a}, \omega_{1a} - \omega_{2a}
\]

The TDA equation is (corresponding):

\[
(E_0 - E_m - 1_{\mu_1 \mu_2})X_{\mu_\mu}^{(m)} + \sum_{\alpha} \left( \chi_\alpha \chi^{(m)} - \chi^{(m)} \chi_\alpha \right) - \chi_\alpha \chi^{(m)} \chi_\alpha = 0
\]

We will now multiply by the 3 CB coefficients from Eq. 1 and sum over all $\mu$'s. Let's look at the spin-dependent CB when multiplying the first potential term. We get

\[
\text{Since } V \text{ is spin-independent, independent:}
\]

\[
\chi^{(m)} \chi^{(n)} \sim \delta_{\omega_{1a} - \omega_{2b}} \delta_{\omega_{1b} - \omega_{2a}} \delta_{\omega_{1a} - \omega_{2a}}
\]

\[
\text{from } S = 0, T = 0
\]

\[
\left( \sqrt{2} \right)^n \left\langle \frac{1}{2} \omega_{1a} \frac{1}{2} \omega_{2b} \left| \frac{1}{2} \omega_{100} \right\rangle \left\langle \frac{1}{2} \omega_{00} \left| \frac{1}{2} \omega_{100} \right\rangle \chi^{(m)} \chi^{(n)}
\]

Thus when this is summed with the CB in Eq. 1 we get $S = 0$ and similarly for the $T$ dependence. In the second potential term the spin-independent CB coefficient go with terms of the potential and onto the wave functions. The
The $l$-dependence is a bit more complicated so see Wolper's notes.

The net result is:

\[
\left[ E_0 - E_{\ell} - (E_{\ell} - E_0) \right] \times \langle \ell \rangle_{\ell_1, \ell_2, \ell_3, \ell_4} + \sum_{\mu} \text{ const}_\mu \times \langle \ell \rangle_{\ell_1, \ell_2, \ell_3, \ell_4} = 0
\]

where

\[
\sum_{\lambda, \lambda'} \langle \mu_{\lambda} \mu'_{\lambda'} | \frac{1}{\ell} \rangle = \frac{1}{2} \left( \delta_{\mu_{\lambda}} \delta_{\mu'_{\lambda'}} \right) \left( \delta_{\lambda, \lambda'} \right) + \frac{1}{4} \sum_{\ell_1, \ell_2, \ell_3, \ell_4} \langle \ell_1 \ell_2 | \frac{1}{\ell} | \ell_3 \ell_4 \rangle
\]

Now let's apply the schematic model: What are the allowed 

\[ \text{proton-} \text{hole states in this coupling scheme.} \]

<table>
<thead>
<tr>
<th>$S$</th>
<th>$L$</th>
<th>$T$</th>
<th>$J$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>L</td>
<td>0</td>
<td>L</td>
</tr>
<tr>
<td>1</td>
<td>L</td>
<td>0</td>
<td>$L-1, L, L+1$</td>
</tr>
<tr>
<td>0</td>
<td>L</td>
<td>1</td>
<td>L</td>
</tr>
<tr>
<td>1</td>
<td>L</td>
<td>1</td>
<td>$L-1, L, L+1$</td>
</tr>
</tbody>
</table>

Clearly the singlet state $[17]$ is split off from the rest since both working elements in (1) contribute.
All states in the \( E_{1S} \) at fixed \( L \) are degenerate, since
the other wavy elements are independent of \( S \) and \( T \).

Calculate first what happens to the \( E_{1S} \)-term. For any
given \( L \), we get 15 degenerate states for each \( n \) obtained
by diagonalizing eq (4). Let's use a very simple
residual interaction: \( V(l, z) = -g \delta(r, \bar{r}_2) \) with \( g > 0 \).

Now:

\[
\sum_{\ell} E_{1S}^{\ell l} = 2 \sum_{\ell} E_{1S}^{\ell l} V_{1S}^{\ell l}
\]

(See Weberbo by definition;
\( \Rightarrow \) separable \( r-l \) interaction).

\[
V_{ab} = (-)^{l_a} \frac{\ell_a}{\sqrt{(2\ell_a + 1)(2\ell_a + 1)!}} \left( \begin{array}{c} 
\ell_a & L & \ell_b \\
0 & 0 & 0
\end{array} \right)
\]

The parameter \( g \) represents the remaining radial integral

\[
g \equiv \frac{\alpha}{4\pi} \int_{0}^{\infty} \text{U(1)} \text{U}(\ell_a \text{U}), \text{U}(\ell_b \text{U}) \frac{dr}{r^2} > 0
\]

Now, since the radial wave functions are peaked
at the nuclear surface for protons in the last few
unoccupied shells and low in the last few filled
shells, and the overlap \( g \) does not change much from
one porrihole pair to the next, we will thus
approximate \( g \) as constant.
Thus:

\[ \sum_{\ell=0}^{\infty} \left( \begin{array}{c} E_\ell - E_n = (E_n - E_0) \end{array} \right) \sum_{\ell=0}^{\infty} \left( \begin{array}{c} \ell \end{array} \right) \chi_{1\ell}(a, b) + \sum_{\ell=0}^{\infty} \left( \begin{array}{c} \sum_{\ell=0}^{\infty} \chi_{1\ell}(a, b) \end{array} \right) \chi_{1\ell}(a, b) = 0 \]

This has a solution.

\[ \text{consistently by } \sum_{ab \neq \ell=0}^{\infty} \frac{\mathcal{N}_{ab}}{E_{ab} - E_n} \]

\[ \Rightarrow \left( \sum_{ab} \mathcal{N}_{ab} \chi_{1\ell}(a, b) \right) + \sum_{ab} \frac{\mathcal{N}_{ab}^2}{E_{ab} - E_n} \left( \sum_{ab} \mathcal{N}_{ab} \chi_{1\ell}(a, b) \right) = 0 \]

\[ \Rightarrow \left( \begin{array}{c} \frac{1}{2} = \frac{\mathcal{N}_{ab}^2}{E_{ab} - E_n} \end{array} \right) \leq \text{eigenvalue equation} \]

For given \( \mathcal{N}_{ab} \), (numbers), given \( \Sigma \) p energies \( E_{ab} \), given overlap \( \frac{1}{2} \), solve for \( E_n \)

\[ \Rightarrow \text{RHS of (x)} \]

If there are \( n \) particles

Stops, all but one eigenvalue

are heated between the \( E_{ab} \)

The remaining state gets pushed up in energy.

The larger the overlaps \( \Sigma \), the larger \( E_{1\ell} \), the energy of this resonant state.
For simplicity let all $p$–$h$ states be degenerate.

\[ E_{nl} = E_0 - E_0 = E_0 \] then the resonant solution

\[ \frac{1}{3} = \frac{1}{E_0 - E_0} \sum (\psi_{nl}^2) \Rightarrow E_0 = E_0 + \frac{1}{3} \sum (\psi_{nl}^2) \]

and the wave function

\[ \chi(nl) = \frac{\sqrt{L}}{(2\pi)^{3/2}} \frac{e^{i \frac{\pi}{4}}}{\sqrt{2n_4}} \left( \frac{\sqrt{\gamma}}{2\pi} \right)^{1/2} \]

is a coherent superposition of all particle–hole states in the box. For the other $n-1$ states in the limit $E_0 = E_0$

we must have:

\[ \sum_{\text{all } \text{states}} \chi(nl)^* \chi(n' l') = 0 \quad \text{for } n' = 0 \] Nondissipative state

Now do we associate "resonant" and "nondissipative"?

Consider how - energy (slow-wavelength) photons absorbed from g.s. Lt is a normal – particle excited state $S_1, S_2, S_3, \ldots$

\[ \sim |P_l^2| \sim |\langle \text{H} | \psi_l^2 | \text{H} \rangle|^2 \sim \left| \sum_{j} \beta_j (k_l) \gamma_j (\mathbf{k}_l) \Delta_j (\mathbf{r}) \right|^2 \]

Transition probability

\[ \text{photon absorption} \]

\[ \text{cross-section} \]

\[ \text{transition probability} \] at \text{wavelength}
Thus for each $L$, the pure supermultiplet pushed up in energy carries the entire transition strength. The other $n-1$ degenerate supermultiplets within the same $L$ have no transition strength at all. This is what is observed for giant resonances. For example,

\[ \text{(G02)} \]

\[ S=0, \quad T=1, \quad L^T=J^T=1; \quad E_{\text{giant}}=10-100\text{MeV} \]

So a broad state with high excitation energy that

\[ \text{carries almost all of the transition strength.} \]
11 Application to a Real Nucleus - $^{16}$O

Many calculations of nuclear spectra starting from realistic single-particle properties and two-nucleon interactions have been carried out (see, for example, Refs.[N64, N65, N66, N67, N68, N69]). It is impossible to summarise all these results here. Rather we present just one example of an attempt to calculate the excited states of a real nucleus. The calculation focuses on the negative-parity $T = 1$ states of $^{16}$O; these are the states excited in inelastic electron scattering at large angles through the large isovector magnetic moment of the nucleon [Eqs.(7.76) and (8.31)]. The calculation is due to Donnelly and Walker in Ref.[N70] (see Ref.[N2]).

One starts with single-particle states of the form $|nljm; \frac{1}{2}m_s\rangle$ which diagonalize the strong spin-orbit force $H_{so} = V_s(r)l \cdot s$. The analysis of Secs.9.10 is readily generalised to this case (Prob.9.3). The ground state of $^{16}$O is assumed to form a closed p-shell. All particle-hole states corresponding to a hole in the p-shell and a particle in the next (2s-1d) oscillator shell are retained (Fig.11.1). The particle-hole configuration energies $\varepsilon_a - \varepsilon_b$ are taken from the neighboring oxygen isotopes as indicated in Fig.11.2. They are shown in Table 11.1.

A non-singular Serber-Yukawa potential fit to low-energy nucleon-nucleon

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Footnote: Note from Eq.(8.16) $\mu = \frac{1}{3}(\Lambda_p + \Lambda_n) + \frac{1}{3}(\lambda_p - \lambda_n)$. Since $(\lambda_p - \lambda_n) \gg (\Lambda_p + \Lambda_n)$ it is the isovector transitions that dominate the tranverse electron scattering cross section.
Table 11.1: Particle-Hole configurations retained in calculation of negative-parity $T = 1$ states in $^{16}$O and configuration energies obtained from neighboring nuclei.

<table>
<thead>
<tr>
<th>Configurations</th>
<th>$e_a - e_b$ (MeV)</th>
<th>States</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(2s_{1/2})(1p_{3/2})^{-1}$</td>
<td>18.55</td>
<td>$1^-, 2^-$</td>
</tr>
<tr>
<td>$(1d_{5/2})(1p_{3/2})^{-1}$</td>
<td>17.68</td>
<td>$1^-, 2^-, 3^-, 4^-$</td>
</tr>
<tr>
<td>$(1d_{5/2})(1p_{3/2})^{-1}$</td>
<td>22.76</td>
<td>$0^-, 1^-, 2^-, 3^-$</td>
</tr>
<tr>
<td>$(2s_{1/2})(1p_{1/2})^{-1}$</td>
<td>12.39</td>
<td>$0^-, 1^-$</td>
</tr>
<tr>
<td>$(1d_{5/2})(1p_{1/2})^{-1}$</td>
<td>11.52</td>
<td>$2^-, 3^-$</td>
</tr>
<tr>
<td>$(1s_{1/2})(1p_{1/2})^{-1}$</td>
<td>16.60</td>
<td>$1^-, 2^-$</td>
</tr>
</tbody>
</table>

scattering is used

$$V(1, 2) = \left[ 1^1 V(r_{12})^1 P + 3^3 V(r_{12})^3 P \right] \frac{1}{2} [1 + P_M(1, 2)]$$

$$1^1 P = \frac{1}{4} (1 - \sigma_1 \cdot \sigma_2)$$  \hspace{1cm} $$3^3 P = \frac{1}{4} (3 + \sigma_1 \cdot \sigma_2)$$

$$V(r_{12}) = V_0 e^{-\mu r_{12}}$$

$$1^1 V_0 = -46.87 \text{ MeV} \hspace{1cm} 1^1 \mu = 0.8547 \text{ fm}^{-1}$$

$$3^3 V_0 = -52.13 \text{ MeV} \hspace{1cm} 3^3 \mu = 0.7261 \text{ fm}^{-1}$$

(11.1)

The calculation employs harmonic oscillator single-particle solutions (Sec. 6) as approximate Hartree-Fock single-particle wave functions with an oscillator parameter $b = 1.77 \text{ fm}$ determined from a fit to elastic electron scattering. The calculated spectrum for $^{16}$O is shown in Fig. 11.3. Also shown in this figure is the spectrum with the spin-dependent forces ($H_{\text{sd}}$ and $V_{1} \sigma_1 \cdot \sigma_2$) turned off. In this case one reproduces the previous model results of Sec. 10.60

The cross section for photoabsorption involves the dipole states with $(J^p, T) = (1^-, 1)$; the comparison of the observed photoabsorption cross section for $^{16}$O with the calculated values (arbitrary overall normalisation) is indicated schematically in Fig. 11.4. The total calculated strength is too high by about a factor of 2.

The use of the current and magnetisation operators in Eqs. (8.12) allows one to compute the electron scattering cross section [Eq. (7.76)] to the discrete levels in Fig. 11.3. The results are compared with the experimentally observed $(e, e')$ spectrum at $\theta = 135^\circ$ and $E_i = 224 \text{ MeV}$ in Fig. 11.5. The solid curve is an estimate at this momentum transfer of the nonresonant background above the threshold for nucleon emission.

60The [16] supermultiplets here are obtained from the spatial states $(2s)(1p)^{-1}$ and $(1d)(1p)^{-1}$, where the total $L$ is indicated with a subscript.
Figure 11.3: Calculated spectrum of $T = 1$ negative-parity excitations of $^{16}$O. Also shown is the calculated spectrum with the spin-dependent forces turned off. From Refs. [N70, N2].

Figure 11.4: Schematic comparison of observed and calculated photoabsorption cross section in the giant resonance region for $^{16}$O.
Figure 11.5: Experimentally observed spectrum of scattered electrons at $\theta = 135^\circ$ and $\epsilon_i = 224$ MeV compared with calculated spectrum for states in Fig.11.3 (arbitrary overall normalization; the integrated areas for the various complexes are compared with theory in the next figure). From Refs.[N2, N71].

The form factors for the various complexes observed in Fig.11.5 are compared with the experimental data (area under the resonance peaks) in Fig.11.6. The theoretical results for the form factors are all too high, and they are reduced in amplitude by approximately the following numerical factors for each of the indicated complexes: \( 2/3 \) (13 MeV); \( 2/3 \) (17 MeV); \( 1 \) (19 MeV); \( 2/3 \) (20.4 MeV); \( 1/\sqrt{2} \) (Coulomb part of giant dipole resonance).

In summary, the shell model provides a basis for understanding the dominant features of the set of negative-parity $T = 1$ particle-hole excitations in this nucleus up to excitation energies of the order of 30 MeV. Linearization of the equations of motion for the collective particle-hole excitations provides a semi-quantitative description of both the location of the levels and the spatial distribution of the transition current densities through which they are excited by the electromagnetic interaction.