Finite nuclei:

So we have been almost successful in describing properties of a nucleus written in terms of two-body non-interactions. We have shown that excluding two-body short-distance correlations are important for realistic potentials with a hard core, the healing distance is short enough to justify independent particle approach almost everywhere else. We will therefore extend now this approach (in nonrelativistic nucleons with two body forces + independent particle approximation or shell model) to describe finite nuclei.

To discuss (finite) nucleus structure the main difference is that we cannot use plane wave basis or single particle basis. The finite extent of matter means that wave functions are strongly modified from their free (so wave) plane wave form. Compare simple harmonic oscillator SHO w. fermions and plane waves.
We want to choose single particle wave functions in a suitable manner so that when expanding eigenstates of $H$

\[ H = \sum_{i=1}^{A} \frac{1}{2} K_i + \sum_{i,j=1}^{A} V_{ij} \]

only few Slater determinants appear, so in the language of second quantization only a few single particle orbitals are involved. This means that in choosing $\phi_n(x)$'s (single particle wave functions) we want to include as much dynamics as possible. We will come back to this later. For now we will assume we can start with a filled Fermi sea of single particle orbitals of finite extent (i.e. a single Slater determinant).
Take the system to be spherically symmetric

with single-particle states:

\[ |\ell s j m_j\rangle \quad ; \quad s = \frac{1}{2}, \; j = |\ell + \frac{1}{2}| \quad (\ast) \]

(we will add isospin later)

\[ |\ell\rangle \equiv |\alpha, m_{\alpha}\rangle \]

Define \(-\alpha\rangle \equiv |\alpha, -m_{\alpha}\rangle \quad (\alpha \text{ stands for } \ell s j m_j)\)

Just like in the case of the nucleon wave, we'll define fields:

\[ \Psi(x) = \sum_{\alpha} \Psi_{\alpha}(x) C_{\alpha} \]

\[ \langle C_{\alpha}; C_{\alpha}^+ \rangle = \delta_{\alpha \alpha} \quad \langle C_{\alpha}; C_{\beta}^+ \rangle = \langle C_{\beta}; C_{\alpha}^+ \rangle = 0 \]

Remember our Hamiltonian in the second quantization representation looks just like before for the 36 nucleon wave, except that the indices \(\nu, \sigma, \tau, \ldots\) which were referring the the plane-waves now are replaced by \(\alpha\)'s and refer to a set of 9 numbers as in \((\ast)\) with the plane waves replaced by a set of wave functions:

\[ \Psi_{\alpha}(x) \quad \chi = \ell, s, j, m_j \quad \text{(isospin later)} \]
I have also introduced $c$'s (and $c^+$'s) instead of $\tilde{c}$'s (and $\tilde{c}^+$'s) used before to describe fermion creation (annihilation) operators. So finally:

$$H = \sum \langle \alpha | t | \beta \rangle c^+_\alpha c_\beta + \frac{i}{\hbar} \sum c^+_\alpha c^+_{\beta'} \langle \alpha | \beta' \rangle V_{\alpha \beta'} c_{\beta'} c_\alpha$$

Our ground state is chosen as:

- States with $E_\alpha \leq E_F$ filled: $| \Psi \rangle = \bigotimes \left| \alpha \right\rangle$
- States with $E_\alpha > E_F$ empty: $\tilde{| \Psi \rangle}$ a ground state.

Again, $E_\alpha$'s are single particle energies:

$$H_0 \Psi_\alpha(x) = E_\alpha \Psi_\alpha(x)$$

Some chosen Hamiltonian to determine $\Psi_\alpha(x)$'s.

Now define new operators $c^+_\alpha$ (if $E_\alpha > E_F$)

- Creates particle above $E_F$.

and $b^+_\alpha = S^{-1}_\alpha c^+_\alpha$ (if $E_\alpha < E_F$) $S^{-1}_\alpha = (-)^{j_{\alpha} - m_{\alpha}}$

- Destructs particle below $E_F$ and creates a hole.

$$c_\alpha = \begin{cases} a_\alpha & \text{if } E_\alpha > E_F \\ S_{\alpha} b^+_\alpha & \text{if } E_\alpha < E_F \end{cases}$$
\[ b_+^{\dagger} 10 \rangle = S_{-\sigma} (c_{-\sigma} 10 \rangle \]

\[ C_\alpha = \theta (\varepsilon_\alpha - \varepsilon_F) a_\alpha + \theta (\varepsilon_F - \varepsilon_\alpha) S_\alpha b_{-\alpha}^+ \]

\[ C_\alpha^+ = \beta (\varepsilon_\alpha - \varepsilon_F) a_\alpha^+ + \beta (\varepsilon_F - \varepsilon_\alpha) S_\alpha b_{-\alpha}. \]

This a canonical transformation \( \Rightarrow \) \( d \{ a_\alpha^+ a_\alpha \} = d \{ b_\alpha^+ b_\alpha \} = 0 \)

\( \alpha \) momentums match.

Why do we need this phase \( S_{-\sigma} \): to get states that transform properly under rotations:

\[ \sum_{\alpha, \alpha'} C_\alpha^+ \langle \alpha | \tilde{J}_0 | \alpha' \rangle = \sum_{\alpha, \alpha', \omega} \langle j_0 | \tilde{J}_0 \tilde{J}_0 \rangle a_0^+ a_{\omega, \omega'} \]

\( \sum \) states are even:

\[ + \sum_{\alpha, \alpha', \omega} \langle j_0 | \tilde{J}_0 \tilde{J}_0 \rangle (-)^{j_0 - \omega} b_{\omega, \omega'} (-)^{j_0 - \omega} b_{\omega, \omega'}^+ \] 

\[ \Rightarrow (-)^{\omega} \sum_{\omega} \left[ \delta_{\omega, \omega'} - b_{\omega}^{\dagger} b_{\omega'} \right] \]

Do \( \delta_{\omega, \omega'} \) first \( \sum_{\alpha, \alpha', \omega} \langle j_0 | \tilde{J}_0 \tilde{J}_0 \rangle \delta_{\omega, \omega'} = \sum_{\alpha < \omega} \sum_{\alpha \geq \omega} \langle j_0 | \tilde{J}_0 \tilde{J}_0 \rangle \]

\[ = \sum_{\alpha < \omega} \sum_{\omega} \omega = 0 \quad \text{(spherical symmetry)} \]
The second term: $\langle j_{m'1} j_{m'} \rangle \Rightarrow \langle j_{m'1} j_{m'} \rangle$

\[
\langle j_{m'1} j_{m'} \rangle = (-)^{m-m'} \langle j_{m'} \rangle \langle j_{m'} \rangle
\]

\[
= (-)^{m-m'} \langle j_{m'} \rangle \langle j_{m'} \rangle
\]

\[
= (-)^{m-m'} \langle j_{m'} \rangle \langle j_{m'} \rangle
\]

\[
= (-)^{2(m-m')} \langle j_{m'} \rangle \langle j_{m'} \rangle
\]

\[
= (-)^{m-m'} \langle j_{m'} \rangle \langle j_{m'} \rangle
\]

This phase will now kill $(-)^{m-m'}$ in front of $b_{m_1}^+ b_{m_2}^+$ term.

\[
\Rightarrow \sum \langle j_{m'1} j_{m'} \rangle a_{m_1}^+ a_{m_2} + \sum \langle j_{m'1} j_{m'} \rangle b_{m_1}^+ b_{m_2}^+
\]

Finally, one can easily prove:

\[
[\sum_{m} b_{m}^+, b_{m}^+] = \sum_{m} \langle \beta | j_{m}^+ | \beta \rangle [b_{m}^+, b_{m}]
\]

\[
= \sum_{m} \langle \beta | j_{m}^+ | \beta \rangle b_{m}^+ b_{m} = \sum_{m} \langle j_{m}^+ | j_{m} | j_{m} \rangle b_{m}^+ b_{m}
\]

\[
\Rightarrow b_{m}^+ b_{m} \text{ is on ITQ to vqth j}_m \text{ and componet}_m \text{ vqth a}_m.
\]

Similar proof for $a_{m}^+ a_{m}$.