Independent Pair Approximation.

We have shown that nonsingular, Senior potential does not
provide saturation in the Fermi gas model. A more
realistic potential with a hard core cannot be used
together with the Fermi gas model approximation.

The potential is too singular and its effects
to short distances has to be taken into account
to all orders. Let then use the following (new)
approximation. We will solve the 2-N wave
function exactly in a core region in the
presence of the Fermi gas of A-2 nucleus.

Let's first rewrite the Schrödinger equation into a
more appropriate form for this approximation:

\[ \left[ T_1 + T_2 + V(r) \right] \Psi(1,2) = E \Psi(1,2) \]  

\( \Psi \) is the

potential energy

for 2N

\( E \)

we will take the effect

of the other nucleons

in a moment. \( \)
Define the complete set of basis states for $N$ as eigenstates of $H_0 = T_1 + T_2$:

$$H_0 \phi_n = (T_1 + T_2) \phi_n = E_n \phi_n$$

Rewrite as an integral equation and expand in terms of $\phi_n$:

$$\Psi(1,2) = \phi_0(1,2) + \sum_{n \neq 0} \phi_n(1,2) \frac{\langle \phi_n | V | \Psi(1,2) \rangle}{E - E_n} \tag{2}$$

where the normalization condition for $\Psi(1,2)$ has been chosen as $\langle \phi_0 | \Psi(1,2) \rangle = 1$ which implies:

$$E - E_0 = \langle \phi_0 | V | \Psi(1,2) \rangle \Rightarrow \langle \phi_1 | V | \Psi(1,2) \rangle = \langle \phi_0 | V | \Psi(1,2) \rangle = (E - E_0) \langle \phi_0 | \Psi(1,2) \rangle \square$$

$$(H_0 - E) \Psi(1,2) = (E_0 - E) \phi_0 + \sum_{n \neq 0} \phi_n(1,2) \frac{\langle \phi_n | V | \Psi(1,2) \rangle}{E - E_n}$$

$$= (E_0 - E) \phi_0 + \sum_{n \neq 0} (E_n - E) \phi_n \langle \phi_0 | \Psi(1,2) \rangle \frac{\langle \phi_n | V | \Psi(1,2) \rangle}{E - E_n}$$

$$= (E_0 - E) \phi_0 + \sum_{n \neq 0} (E_n - E) \langle \phi_0 | \Psi(1,2) \rangle \frac{\langle \phi_n | V | \Psi(1,2) \rangle}{E - E_n}$$

$$= -\phi_0 \langle \phi_0 | \Psi(1,2) \rangle - \sum_{n \neq 0} \phi_n \langle \phi_0 | \Psi(1,2) \rangle \frac{\langle \phi_n | V | \Psi(1,2) \rangle}{E - E_n}$$

Completeness of $\phi_n$'s.
The wavefunction basis states are:

\[ \Psi_n(1/2) = \frac{1}{\sqrt{2}} e^{i \vec{P}_1 \cdot \vec{x}_1} \frac{1}{\sqrt{2}} e^{i \vec{P}_2 \cdot \vec{x}_2} \]  

We will now treat the interaction as spin dependent so that we can consider particles with spin \( \uparrow \) and \( \downarrow \) as distinguishable. So if (4) corresponds to \( \Psi_A \) we need not worry about our symmetries (there are no exchanges of nucleons, if particles are distinguishable).

Introduce as usual the total and relative coordinates:

\[ \vec{P} = \vec{P}_1 + \vec{P}_2, \quad \vec{R} = \frac{1}{2} (\vec{R}_1 - \vec{R}_2); \]
\[ \vec{R} = \frac{1}{2} (\vec{R}_1 + \vec{R}_2), \quad \vec{x} = \vec{R}_1 - \vec{R}_2 \]

Since the nuclear wave is uniform \( \vec{P} \) is conserved and we can separate c.m. motion from the full wave function:

\[ \Psi(1/2) = \frac{1}{\sqrt{2}} e^{i \vec{P} \cdot \vec{x}} \frac{1}{\sqrt{2}} \Psi_{1/2}(\vec{R}) \]

The relative wave function will depend on \( \vec{x} \) since one has effects of the residual A-2 nucleons are taken into account. This will introduce a dependence on \( \vec{x} \) in motion with respect to the remaining A-2 nucleons.
We will take the effect of these A-2 nucleons by simply restricting the sum over intermediate states in Eq. 2 to 2 to also exclude these 2- N states which are occupied by the A-2 nucleons but restrict the momenta of the two nucleons in the intermediate states to be below Fermi momentum.

\[
\sum_{n=0}^{2} \quad \sum_{k_F > k_F} \quad \text{etc.}
\]

\[
\Psi_{\text{in}}(\mathbf{r}) = e^{-i\mathbf{k'}\mathbf{r}} + \int \frac{d^3 t}{i\hbar} e^{i\mathbf{k} \cdot \mathbf{r}} \int d^3 y e^{-i\mathbf{k} \cdot \mathbf{y}} \phi_\mathbf{k} \Psi_{\text{in}}(\mathbf{y})
\]

\[
2\mu (E - E_0) - 2\mu \Delta E = k^2 - k^2 = \frac{1}{\sqrt{-i\partial^2}} \int d^3 x e^{-i\mathbf{k} \cdot \mathbf{x}} \phi_\mathbf{k} \Psi_{\text{in}}(\mathbf{x})
\]

Where \(\phi_\mathbf{k}(\mathbf{r}) = 2\mu \Psi_{\text{in}}(\mathbf{r})\)

\[
E = \frac{\mathbf{p}^2}{2m_{\text{total}}} + \frac{\mathbf{R}^2}{2\mu} - \text{baryon mass} = \frac{\mathbf{p}^2}{2\mu} + \frac{k^2}{c^2(\mu\hbar)}
\]

\[
\mathbf{p}^2 \Delta \mathbf{p} > k_F^2
\]

\[
F: \quad |\mathbf{p}^2 - \mathbf{p}'^2| < k_F^2
\]
This is known as the Bethe–Goldstone equation.

We will next solve the BG equation, and calculate the energy shift at each single pair to all orders in $V$ in the presence of other nucleons ($\text{Pauli blocking} \rightarrow R$).

Then we will simply add all the energy shifts of all the pairs to get total energy $E_0$ of the nucleon matter.

$\Rightarrow$ This is known as the independent pair approximation.

This is only an approximation. Better then the Fermi gas model, but we are still ignoring $3$-body and higher body effects:

\[
\begin{array}{c}
\text{V} \\
\text{p}_1 \\
\end{array}
\]

Furthermore, the BG equation is not self-consistent; either it ignores interactions between the $2N$'s in a pair and the medium. This can be included to some extent using the effective mass:

\[
E^0_N = \frac{k^2}{2m} \Rightarrow E^0_N = E^0_N + V(n) \approx \frac{k^2}{2m^*} + U_0
\]

Since energy denominator involves energy differences $U_0$ couples.

Now the effective potential $\nabla^2(r) = 2m^* V(n)$ with $m^* = \frac{m^x}{c^2}$.

$m^*$ should be calculated self-consistently: $H^* \Rightarrow \psi_{2N} \Rightarrow U(n) \Rightarrow m^*$.

Since $k^2$ depends on $V$ (though the normalization condition), the resulting equation is non-linear.
Bethe Goldstone equation: discussion:

1) BGE is like an ordinary scattering equation except the integral over intermediate states is restricted to unoccupied states above Fermi energy.

2) It is an integral equation to \( n \to \infty \) to all orders so can bound repulsive core (Ryutin potentials).

3) We get the energy shift/pair as proportional to \( \sqrt{V} \). However, when we sum over all pairs ("independent pair approximation") the sum cancels \( V \) and produces a finite answer. Remember we have derived BGE for distinguishable particles (opis A6). For identical particles our loss to configuration.

4) This effect gives the exact energy shift up to \( O(V^2) \) in perturbation theory:

\[
E_2 = \begin{pmatrix} \cdots \end{pmatrix} - \begin{pmatrix} \cdots \end{pmatrix}
\]

At higher orders it includes a subset of (Pauli) exclusion.
(15572)

5) We improve "softening" of BSE by replacing free kinetic energies $E^0_k$ by effective due to simple point interactions with nuclear matter $U$. For interaction $U$ we have:

$$E^0_k \rightarrow E_k = E^0_k + U(h)$$

and use effective mass approximation:

$$U(h) \approx U_0 + \frac{m^2}{2m^*} = \Rightarrow E_k = \frac{p^2}{2m^*}$$

$$m^* = \frac{m}{1+U}$$