Lab #2: Electrical Measurements II—AC Circuits and Capacitors, Inductors, Oscillators and Filters

**Goal:** In circuits with a time-varying voltage, the relationship between current and voltage is more complicated than Ohm’s law for resistance. In this laboratory you will study time-varying electrical signals to understand the current-voltage (I-V) relationship for capacitors and inductors, see how resistors, capacitors and inductors combined lead to the concept of a complex impedance, and build passive low-pass and high-pass filters (circuits that have high impedance for high frequencies and low frequencies respectively) using RC and LC circuits.

**Note:** As usual, Wikipedia has some excellent articles on LR and RC circuits and the concepts of phase shifts and complex impedance. You may wish to refer to these articles as you work through your lab.

**Equipment:** non-polar capacitors (2 μF and 0.1 μF), inductor (800 mH), resistors, function generator to produce time-varying voltages, DMMs, a digital oscilloscope (DS0), Proto-Board, BNC coaxial cables, BNC tee junctions.

1. Measuring Capacitance

A capacitor, as its name implies, stores electrical charge. The voltage across a capacitor of capacitance $C$ relates to the charge stored in the capacitor as:

$$V = \frac{Q}{C}.$$  \hspace{1cm} (eq. 1)

If you charge a capacitor with a power supply of voltage $V$ and then disconnect it, the charge will remain in the capacitor for a considerable amount of time. If you then connect a resistor with resistance $R$ across the charged capacitor, the charge in the capacitor will discharge through the resistor, producing a current $I = \frac{V_{\text{Capacitor}}}{R}$. Since the charge on the capacitor decreases as:

$$\frac{dQ}{dt} = -I = -\frac{V_{\text{Capacitor}}}{R}.$$ \hspace{1cm} (eq. 2)

Substituting for $Q$ from equation 1:

$$\frac{dV_{\text{Capacitor}}}{dt} = -\frac{V_{\text{Capacitor}}}{RC},$$ \hspace{1cm} (eq. 3)

So the voltage across the capacitor decays exponentially with a time constant $\tau = RC$, i.e.:

$$V_{\text{Capacitor}}(t) = V_0 e^{-\frac{t}{RC}},$$ \hspace{0.5cm} with time constant $\tau = RC$. \hspace{1cm} (eq. 4)

Remember that $1\Omega \times 1F = 1s$. 1F is a very large capacitance. Most capacitors have capacitances in the range of μF. If the resistance $R$ across the capacitor is just the internal resistance of your DMM voltmeter (of the order of $10 - 20 \Omega$) and the capacitance of
the capacitor is about 2 μF, the time constant is about 20 – 40 s, long enough for you to record ‘by hand’ while the discharge is taking place.

As with resistors, the nominal capacitance of a capacitor is often quite different from its actual capacitance. We can use the result in equation 4 to measure the capacitance of an unknown capacitor. First, set your DMM to the 20 volt DC range and measure its internal resistance as in the previous lab. Connect the DMM across the capacitor as shown in Figure 1. **Note that the DMM is in parallel with the capacitor.** Then charge the capacitor to 15 – 20 V by temporarily connecting the Proto-Board’s power supply across the capacitor. Disconnect the power supply and record the voltage on the voltmeter every 10 s.

Improve your measurement by using a more sensitive DMM DC voltage range. Remember to measure the DMM internal resistance for each voltage range (you can use the circuit from Lab 1, part 1 to measure the internal resistance of the voltmeter).

Questions:
1) Plot the voltage versus time on a semi logarithmic scale and determine the RC time constant \( \tau \) from the slope of the \( V(t) \). Estimate the uncertainty of your measurement of RC. Where does this uncertainty come from? It may have several contributing components.
2) Deduce \( C \) from \( \tau \) using equation 4 and estimate the error in your measurement of \( C \) using error propagation.
3) What effect did the more sensitive DMM range have on your error in your measurement of \( C \)?

**Figure 1:** Circuit for measuring capacitance using the time constant of an RC circuit.
Measuring AC voltage

For alternating currents, the amplitude of the voltage is somewhat ambiguous (see Figure 2). The most common ways to characterize AC voltage are Peak-to-Peak ($V_{\text{peak-to-peak}} = V_{\text{max}} - V_{\text{min}}$) and RMS voltage ($V_{\text{RMS}} = \sqrt{\frac{1}{T} \int_0^T (V(t))^2 dt}$). For a sine wave of amplitude $A$ and frequency $\omega$, $V(t) = A \sin(2\pi\omega t)$, $V_{\text{peak-to-peak}} = 2A$, and $V_{\text{RMS}} = \sqrt{\frac{1}{T} \int_0^T V(t)^2 dt} = \sqrt{\frac{1}{T} \int_0^T A^2 \sin^2(2\pi\omega t) dt} = \frac{A}{\sqrt{2}} = \frac{V_{\text{peak-to-peak}}}{2\sqrt{2}}$. Note that for a waveform other than a sine wave, the relationship between peak-to-peak and RMS voltage will differ, e.g. for a square wave $V_{\text{RMS}} = \frac{V_{\text{peak-to-peak}}}{4}$, for a triangle wave, $V_{\text{RMS}} = \frac{V_{\text{peak-to-peak}}}{2\sqrt{3}}$ (see https://en.wikipedia.org/wiki/Root_mean_square for more details).

![Figure 2: Different ways of specifying the amplitude of an AC voltage.](image-url)
Set your function generator to produce a 20Hz sine wave with a 10V peak-to-peak amplitude. Set your DMM to the AC 20V range. Connect the DMM and the oscilloscope in parallel across the output of the sine wave produced by the function generator as in Figure 3. See the handout and refer to the introduction in class to learn how to work with the oscilloscope. Make sure to set the oscilloscope to a DC voltage range with the cursors set to measure peak-to-peak voltage (they can also be set to measure RMS voltage) and adjust the frequency for each measurement so you see 2-5 periods of oscillation on the screen. What is the RMS (root-mean-squared) voltage corresponding to this peak-to-peak voltage? (Measure the peak-to-peak voltage with the oscilloscope and the RMS voltage with the DMM and compare your results.) The oscilloscope also has a measurement mode which you can use to do the same calculation. Make sure your results are comparable and report any differences. Repeat the measurement on both instruments for frequencies up to 100 kHz (use frequencies of 20Hz, 50Hz, 100Hz, 200Hz, 500Hz, 1kHz, 2kHz, 5kHz, 10kHz, 20kHz, 50kHz, 100kHz), i.e., a log-scale of frequencies.

Repeat your measurement of $V_{\text{RMS}}$ and $V_{\text{peak-to-peak}}$ at 20Hz and 200Hz for a triangle wave and a square wave (use the appropriate settings on your function generator and be sure to sketch the wave-forms in your lab book).

Questions:
1) What are the main sources of error in your measurement? Estimate their relative error contributions.
2) At what frequency do the readings on each instruments begin to differ? Why? Which of the two is unreliable at high frequency?
3) For the square and triangle waves, what are your measured relationships between $V_{\text{RMS}}$ and $V_{\text{peak-to-peak}}$? Do they agree with the theoretical values?

Figure 3: Circuit to compare peak-to-peak and RMS voltages.

3 Capacitor in a voltage divider circuit (complex impedance)
As discussed in the handouts, the current and voltage in a circuit can have a more complicated relationship than for a DC circuit. The current in a capacitor relates to the voltage across the capacitor as:

\[ I(t) = C \frac{dV(t)}{dt}. \]  
(eq. 5)

We can also express this relationship (for a sine wave applied voltage) as a complex impedance:

\[ Z = \frac{1}{i\omega C}. \]  
(eq. 6)

We can use this relationship to measure the capacitance of an unknown capacitor and to understand the phase shift and amplitude attenuation in the voltage across the capacitor. We define the phase shift \( \phi = 2\pi \Delta t \omega \) and the amplitude attenuation as \( \delta A = \frac{V_1}{V_0}. \) Note that the phase shift can be positive or negative and that there is an ambiguity of sign for large phase shifts, if \( \Delta t \gg \frac{1}{2\omega}. \) In Figure 4, the orange curve is ahead of the blue curve in time, so \( \phi > 0. \)

![Figure 4: Phase and attenuation between two sine-wave voltages. On your oscilloscope, use the red arrow to measure the peak-to-peak time shift. The orange curve is ahead of the blue curve in time, so \( \phi > 0. \)](#)

To help you analyze the circuit, if the resistor and capacitor are in series (as in Figure 4), the current through the resistor and the current through the capacitor must be equal at all times. In addition, the sum of the voltage across the capacitor and the voltage across the resistor must be voltage produced by the function generator at all times.

Assemble the circuit shown in Figures 5 and 6 in order to measure the complex impedance of a capacitor. Use the blue 0.1\( \mu \)F capacitor and a 1 k\( \Omega \) resistor. Measure the value of \( R \) accurately with a DMM before you put it in the circuit. For a log-scaled range of...
frequencies from 20 Hz to 100 kHz (use frequencies of 20 Hz, 50 Hz, 100 Hz, 200 Hz, 500 Hz, 1 kHz, 2 kHz, 5 kHz, 10 kHz, 20 kHz, 50 kHz, 100 kHz), measure both the amplitudes of $V_0$ and $V_1$, the phase shift $\phi$ of $V_1$ (w.r.t. $V_0$), and the frequency. Remember that if

$$V_0(t) = \text{real}(A_0 e^{i\omega t})$$

then

$$V_1(t) = \text{real} \left( V_0 \frac{R}{R + \frac{1}{i\omega C}} \right),$$

(eq. 7)

and that,

$$\phi = \frac{2\pi \Delta t}{\tau} = 2\pi \Delta t f = \Delta \omega,$$

(eq. 8)

where $\tau$ is the period of oscillation. Use the oscilloscope cursors and measurement features to make these measurements (see the oscilloscope information sheet).

Questions:

1) What are the main sources of error in your measurement? Estimate their relative error contributions.

2) Assuming that the voltage output from the function generator is $V(t) = A \sin(2\pi\omega t)$, calculate theoretically the voltages and currents across the resistor and capacitor as a function of time? What is the phase shift between the voltage across the function generator $V_0$ and the voltage across the resistor $V_1$?

3) Calculate the value of $C$ by fitting your experimental amplitude and phase data vs. frequency to the theoretical values. In both plots use $\omega$ for the x-axis and $V_1$ for the y-axis for the amplitude plot and $\phi$ for the y-axis for the phase plot. You can do this fit in Excel or using any program you like (make sure to do a Least Squares fit with $C$ as a fitting parameter. We will supply sample Excel code on-line. What is your inferred value of $C$? What is the uncertainty in $C$?

4) Putting this value of $C$ back into your theoretical relations, compare your experimental results for amplitude and phase to the theory and plot them together on a pair of graphs. Do the theoretical and experimental values agree within error? Does the agreement differ for low and high frequencies?

5) Like a voltmeter, an oscilloscope has an internal resistance $R_{osc}$ (it is approximately 1 MΩ for this oscilloscope). What is the expected change in your measured $V_1$ due to this resistance? Is this difference significant compared to the other errors in your calculation (give numbers)? Does the relative effect depend on frequency? The oscilloscope also had an internal capacitance of 20 pF as well? Is this capacitance significant? If so, what is its effect on your measurement? At what frequency is $\left| \frac{1}{i\omega C} \right| \approx R_{osc}$?

6) For coaxial cable of type RG-58, the capacitance is 25 pF/foot (the units are strange but cables are usually specified in this mixture of SI and imperial units!). What is the complex impedance of the cable you are using? How does it compare in magnitude to the resistors and capacitors you are using in your circuit? Is this cable capacitance significant?
**Figure 5:** Circuit to observe the effect of a capacitor on the phase and amplitude of a sine-wave signal as a function of frequency.

**Figure 6:** Photograph of apparatus for RC circuit. Note that on the oscilloscope, channel 2 (blue) is shifted to the left of channel 2 (yellow) in time, i.e., $V_1$ is ahead of $V_0$, so the phase shift is positive.
4 RC and LC low-pass and high-pass filters

Because of the frequency dependence of the AC voltage-current relation of capacitors and inductors, we can construct voltage divider circuits that pass some frequencies and block others. Here we present four low- and high-pass filters made either from resistors and capacitors, or resistors and inductors (Figure 7). The circuit you built in Section 3 was a high-pass filter. In this section, you will study the four simple types of these circuits. You can combine these to make more complex filters like band-pass filters or filters with varying steepness in frequency.

For any filter, the ability to reduce the amplitude of undesired frequencies is called the attenuation of the filter. The decibel unit is often used to describe the amount of attenuation produced by the filter. The gain or loss (attenuation) in decibels is defined by:

\[
N_{dB} = 20 \log_{10} \left( \frac{V_{out}}{V_{in}} \right) \tag{eq. 9}
\]

The frequency at which the input is attenuated by 3dB (decibel), or reduced to 70.7\%, is called the cutoff frequency, \( f_c \). At this frequency the power output is about half that for the unfiltered signal. For the passive filters in Figure 7, \( N_{dB} \) can never be positive, because \( V_{out} \) can never be greater than \( V_{in} \). Remember that \( V_{out} \) and \( V_{in} \) in equation 9 are real amplitudes.

Repeat the amplitude measurements in Part 3 for each of the circuits shown in I, II and IV (you already studied III in Part 3). Follow the design in Figure 5 but substitute the resistors, capacitors and inductors called for in Figure 7. To build circuit I, use the same resistor and capacitor as in Part 3. To build circuits II and IV, use the 800 mH inductor in the plastic housing and a 1 k\( \Omega \) resistor (remember to measure the resistance \( R \) before putting it in the circuit). Over a log-scale of frequencies from 20Hz to 100kHz (as in Part 3), measure the attenuation \( \frac{V_0}{V_1} \).

Extra Credit: Measure the phase difference for circuits II and IV.

Questions:
For each of your three circuits, repeat your theoretical analysis of the attenuation in part 3. Then compare your theory to experiment.
1) What are the main sources of error in your measurement? Estimate their relative error contributions.

2) For I, use a least squares fit to find $C$. You can either use Excel or any statistics program you prefer. Is the value the same as in Part 3 within error? As before, plot the theoretical curves for amplitude for this value of $C$ on the same plots as your experimental data and compare their agreement as a function of frequency. Explain your results.

3) For II and IV use a least squares fit between your theory to your experimental data to determine $L$. As before, plot the theoretical curves for amplitude for this value of $L$ on the same plots as your experimental data and compare their agreement as a function of frequency. The complex impedance for an inductor is $Z = i\omega L$. Calculate the theoretical and experimental $Z$ and compare your results.

4) In contrast with the $RC$ circuit, your measurements in II and IV will not match the theory very well because the value of the inductance changes with frequency. Assume that the theory is exact and that the inductance changes with frequency $L(\omega)$ to determine the implied inductance at each frequency for II and IV. Is the implied inductance the same for circuits II and IV within error? What are some reasons that the apparent inductance could be frequency dependent?

5) The cutoff frequency for an RC circuit should be $f_c = RC$, and for an RL circuit $f_c = \frac{R}{L}$. Measure the cutoff frequency from your amplitude vs. frequency plots and compare to the theoretical values. Do they agree within your error? Explain.

6) **Extra Credit:** Repeat your analysis for the phase relationships you measured for the RL circuits and comment on the agreement between your experiment and theory. **Note**—this can be a difficult analysis. You can look up the Kramers-Kronig Theorem and ask Mike for some extra notes. Stop, if it becomes confusing or too time-consuming,