Indiana University  
Physics P391: Modern Physics  
Homework #3 (Due Monday 2/2/15)

Reading: Harris, 2.7-2.10

1. Two pole vaults, Bill and Jill, are running on a track in opposite directions, each going with speed of 0.700c and carrying their poles horizontally, aligned with their direction of motion. Both poles have a proper (rest) length of 7.50 meters.

(a) What is the apparent length of the poles to a spectator sitting next to the track?
(b) From Bill’s perspective, what is the apparent length of Jill’s pole?
(c) As Jill reaches the end of her run, she plants her pole and vaults so that the pole becomes vertical. According to Bill, who is still running, what is the length of Jill’s pole in this orientation?

2. Two spaceships, each of proper length 100 m, pass near one another heading in opposite directions. If an astronaut at the front of one ship (“A”) measures a time interval of 2.50 × 10⁻⁹ s for the second ship (“B”) to pass by, then

(a) What is the relative velocity of the space ships?
(b) What time interval is measured on ship A between the front of ship B passing by by the front of ship A to its passing by the rear of ship A?

3. Consider a wire carrying a current. Physically this means there are positively charged ions at rest in the laboratory, while electrons move with an average "drift speed" vd (typically a few mm/sec). The wire is electrically neutral, meaning that the linear charge density \( \lambda_{+} \) of the (stationary) ions is equal and opposite that of the electrons (\( \lambda_{-} \)). Will the wire still look neutral to an observer that moves along it at rest with respect to the electrons? What is the net charge density \( \lambda_{w} = \lambda_{+} + \lambda_{-} \) in this frame (in terms of \( \lambda_{+} \) and \( \lambda_{-} \))? Comment on whether the electric and magnetic fields are nonzero according to the observer (i) in the lab frame and (ii) in the frame moving with the electrons.

4. (a) Consider a rotated coordinate system in which the x and y axes are rotated clockwise by an angle \( \theta \) to become \( x' \) and \( y' \). Show that \( x' = x \cos \theta - y \sin \theta \) and \( y' = x \sin \theta + y \cos \theta \).
(b) Show that the Lorentz transformation can be written as: 

\[
\begin{align*}
x' &= x \cos \phi - ct \sinh \phi \\
y' &= y \cosh \phi - x \sinh \phi
\end{align*}
\]
where \( \tan \phi = v/c \), with \( v \) being the velocity (in the x direction) of the frame \( F' \) w.r.t. frame \( F \). This suggests that the Lorentz Transformation can be thought of as a sort of "rotation" in a four-dimensional "space-time".

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The speed of individual electrons is dominated by their thermal velocity, \( \sim 10^6 \) m/sec, but in random directions, so it averages to zero. The drift velocity is the residual effect due to the applied electric field.
c) Dimensions perpendicular to the direction of relative motions between frame is the same \( \Rightarrow L' = L \) proper = 7.5 m

2) a) 

According to \( A' \), it takes \( \tau = 2.5 \times 10^{-6} \) s for \( B' \) to pass by length \( L' \) of \( B' \) is a moving length, and therefore Lorentz contracted according to \( A' \):

\[ L_B = \frac{1}{\sqrt{\alpha}} L \text{ proper} \]

Hence,

\[ \alpha_B = \frac{L_B}{\Delta t} = \frac{1}{\sqrt{\alpha}} \frac{L \text{ proper}}{\tau} \]

\[ \Rightarrow \frac{v_B^2}{c^2} = \left[ 1 - \left( \frac{L_B}{c\tau} \right)^2 \right] \frac{L \text{ proper}}{c^2 \tau^2} \]

\[ \Rightarrow \left( 1 + \alpha_B^2 \right) \frac{v_B^2}{c^2} = \alpha_B^2 \Rightarrow v_B = \sqrt{\frac{\alpha}{1 + \alpha}} c = \frac{1}{\sqrt{1 + 0.132}} c \]

\( \alpha = \frac{L \text{ proper}}{c \tau} = \frac{100 \text{ m}}{3 \times 10^8 \text{ m/s} \times (2.5 \times 10^{-6} \text{ s})} = 0.132 \) \( c \)

\[ \Rightarrow v_B = (0.132)(3 \times 10^8 \text{ m/s}) = 3.96 \times 10^7 \text{ m/s} \]

b) Event 1: Front of \( B' \) passing front of \( A' \)

Event 2: \( \ldots \) rear \( \ldots \)

\[ t_2 - t_1 = \frac{x_2 - x_1}{v_B} \]
$x_2 - x_1$ : distance travelled = proper length of ship $A$

$\Rightarrow t_2 - t_1 = \frac{100 \text{ m}}{3.96 \times 10^5 \text{ m/s}} = 2.53 \times 10^{-6} \text{ s}$

In frame $F$, with current $I$ flowing to the left (recall that current is defined as flow of positive charge, by convention, even though it is the negatively charged conduction electrons that are moving), consider length $l$ of wire:

In this frame, the wire is at rest: Conduction electrons are moving at the drift velocity $\bar{v}_d$, giving rise to current $I$.

The positively charged atoms/molecules with which these mobile, loosely-bound valence electrons are associated, remain stationary.

The wire is neutral: $\lambda_{+/+} = \frac{\text{total charge}}{\text{length}} = 0$. Find linear charge densities for electrons and ions:

$\lambda^- = \frac{N_e(-e)}{L}$

$N_e$: number of conduction electrons in length $L$ per $e$: fundamental charge ($= 1.6 \times 10^{-19} \text{ C}$)

$\lambda^+ = \frac{N_p(+e)}{L}$

$N_p$: number of ions in length $L$ per $+e$ wire.

Now $N_+ = N_+ = N \Rightarrow \lambda^+ = \frac{N e_+}{L} = \lambda^-$

$\lambda_{+/+} = \lambda^+ + \lambda^- = 0$
In frame (F) moving with the conduction electrons, i.e., moving w.r.t. (E) at \( \vec{v} = \vec{v}_d = \vec{V}_d \), electrons will be at rest and ions will be moving at \(-\vec{v}_d\). We must find the new charge density \( \lambda' \). First \( \lambda' \):

\[
\lambda' = \frac{N'(\epsilon e)}{L'} \quad \text{Note: charge is invariant under change in coordinate frames.}
\]

\[
N' = N \quad \# \text{ of electrons in given segment of wire doesn't change}
\]

\[
\Rightarrow \lambda' = -\frac{N_e}{L'}
\]

What is \( L' \), if \( L \) is the length of this segment of wire in the lab? According to observers in (F), \( L' \) is the proper or rest length over which \( N \) electrons are distributed, so

\[
L' = \frac{\lambda'}{\lambda'} = \frac{x_v}{\sqrt{1-(\frac{v}{c})^2}}
\]

\[
\Rightarrow \lambda' = -\frac{N_e}{\sqrt{1-(\frac{v}{c})^2}}
\]

For the positive ions,

\[
\lambda_+ = \frac{N_+(\epsilon e)}{L'} = \frac{N_e}{L'}
\]

\[
N_+ = N_+ = N
\]

However, it is the length \( L \) in (E), where the ions are at rest, that constitutes the rest length over which \( N \) ions are distributed, so

\[
L = \frac{\lambda_+}{\lambda_+}
\]

\[
\Rightarrow \lambda_+ = \frac{N_e}{L} = \frac{x_v}{\sqrt{1-(\frac{v}{c})^2}}
\]

So the total charge density in this frame is

\[
\lambda_{\text{tot}} = \lambda_+ + \lambda_+ = \left(\frac{x_v}{\sqrt{1-(\frac{v}{c})^2}}\right) \frac{N_e}{L} \neq 0
\]

In (F), the wire is no longer neutral, but positively charged. While in (E), there was only a \( \vec{B} \) field present due to the current \( I = (\vec{v}_d, \lambda_+) \), in (F) it is changed and there is a magnetic field, \( \vec{B}' \) (due to current \( I' = \vec{v}_d, \lambda_+ \)) and an electric field, \( \vec{E}' \), due to \( \lambda_+ \).

As you will see in P231/232, \( \vec{E} \) and \( \vec{B} \) transform as:

\[
\vec{E}' = \kappa \vec{E}, \quad \vec{B}' = \kappa \vec{B}
\]

\[
\vec{E}'_+ = \kappa (\vec{E}_+ + \vec{v} \times \vec{B}) \quad \vec{B}'_+ = \kappa (\vec{B}_+ - \vec{v} \times \vec{E})
\]

where \( \kappa \) refers to the direction of motion of (F) w.r.t. (E), given by \( \vec{v} \).
For example, since \( \mathbf{E}_0 = 0 \), \( |\mathbf{B}| = \frac{\mu_0 I}{2\pi s} \), for infinitely long wire

\[ \mathbf{E}' = \mathbf{E}_0' = \mathbf{x}_y \times \mathbf{B} = \mathbf{x}_y \cdot \mathbf{E}_0 \cdot \mathbf{x}_y \times \frac{\mathbf{B}}{2\pi s} \]

\[ = \mathbf{x}_y \cdot \mathbf{E}_0 \cdot \mathbf{x}_y \times \frac{\mathbf{B}}{2\pi s} \]

\[ = \mathbf{x}_y \cdot \mathbf{E}_0 \cdot \mathbf{x}_y \times \mathbf{B} \]

Now, \( \lambda' = \frac{\mathbf{x}_y \cdot \mathbf{E}_0 \cdot \mathbf{x}_y \times \mathbf{B}}{\lambda} \)

Recall that the electric field of an infinitely long charged wire is

\[ |\mathbf{E}'| = \frac{1}{4\pi\varepsilon_0} \cdot \frac{1}{\lambda'} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{1}{\lambda'} \cdot \frac{\mathbf{B}}{c} \cdot \mathbf{x}_y \cdot \mathbf{B} \]

Comparing (1), obtained from transformation equations, using \( E' \) in (2), and (3) obtained directly from charge on wire in (1), we find that they are the same, using

\[ (\varepsilon, c^2) = \left(\varepsilon_0, \frac{1}{\mu_0} \right) = \mu_0 \]

\[ \text{b) Recall properties of hyperbolic functions} \]

\[ \sinh \phi = \frac{e^\phi - e^{-\phi}}{2} \]

\[ \cosh \phi = \frac{e^\phi + e^{-\phi}}{2} \]
Can show

\[
\cosh^2 \phi - \sinh^2 \phi = \left( \frac{e^\phi + e^{-\phi}}{2} \right)^2 - \left( \frac{e^\phi - e^{-\phi}}{2} \right)^2
\]

\[
= \frac{e^{2\phi} + e^{-2\phi} + 2 - (e^{2\phi} + e^{-2\phi} - 2)}{4}
\]

= 1

Setting \( \beta = \tanh \phi \), \( \gamma = \frac{1}{\sqrt{1-\beta^2}} \),

\[
= \frac{1}{\sqrt{1 - \tanh^2 \phi}} = \frac{\cosh \phi}{\sqrt{\cosh^2 \phi - \sinh^2 \phi}}
\]

Can write Lorentz transformations as:

\[
\begin{align*}
    x' &= x - \beta x ct \quad \Rightarrow \quad x' &= \cosh \phi x - ct \left( \tanh \phi \cdot \cosh \phi \right) \\
    ct' &= \gamma ct - \beta x \quad \Rightarrow \quad ct' &= \cosh \phi ct - x \left( \tanh \phi \cdot \cosh \phi \right)
\end{align*}
\]

or

\[
\begin{align*}
    x' &= x \cosh \phi - ct \sinh \phi \\
    ct' &= -x \sinh \phi + ct \cosh \phi
\end{align*}
\]
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3. Consider a wire carrying a current. Physically this means there are positively charged ions at rest in the laboratory, while electrons move with an average “drift speed” \(v_d\) (typically a few \(mm/sec\))^1. The wire is electrically neutral, meaning that the linear charge density \(\lambda_{(+)}\) of the (stationary) ions is equal and opposite of that of the electrons \(\lambda_{(-)}\). Will the wire still look neutral to an observer that moves along it at rest with respect to the electrons? What is the net charge density \(\lambda'_{(+)} + \lambda'_{(-)}\) in this frame (in terms of \(\lambda_{(+)}\) and \(v_d\))? Comment on whether the electric and magnetic fields are nonzero according to the observer (i) in the lab frame and (ii) in the frame moving with the electrons.

4. (a) Consider a rotated coordinate system in which the \(x\) and \(y\) axes are rotated clockwise by an angle \(\theta\) to become \(x_R\) and \(y_R\). Show that \(x_R = x \cos \theta - y \sin \theta\) and \(y_R = y \cos \theta + x \sin \theta\).
(b) Show that the Lorentz transformation can be written as: \(x' = x \cosh \phi - ct \sinh \phi\) and \(ct' = ct \cosh \phi - x \sinh \phi\), (and \(y' = y\) and \(z' = z\)), where \(\tanh \phi = v/c\), with \(v\) being the velocity (in the \(x\) direction) of the frame \(F'\) w.r.t. frame \(F\). This suggests that the Lorentz Transformation can be thought of as a sort of “rotation” in a four-dimensional “space-time”.

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